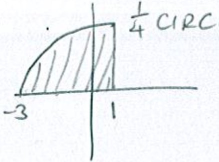
Find the value of c guaranteed by the Mean Value Theorem for Integrals for $f(x)=\sqrt{16-(x-1)^2}$ on $[-3,1]$. **SCORE: _____ / 20 PTS**

$$f_{AVE} = \frac{1}{1--3}\int_{-3}^{1}\sqrt{16-(x-1)^2}\,dx \quad 6$$

$$= \frac{1}{4}\cdot\frac{1}{4}\pi 4^2 \quad 3$$

$$= \pi \quad 2$$



$\frac{1}{4}$ CIRCLE

$$\sqrt{16-(c-1)^2} = \pi \quad 4$$

$$16-(c-1)^2 = \pi^2$$

$$(c-1)^2 = 16-\pi^2$$

$$c-1 = \pm\sqrt{16-\pi^2}$$

$$c = 1 \pm \sqrt{16-\pi^2} \quad 3$$

$$c = 1-\sqrt{16-\pi^2} \in (-3,1)$$

2

After the previous focus group rejected their last product, the company which manufactures plant-based foods changed the recipe for their synthetic meat product and conducted another focus group. The members of this focus group were each given a 4 ounce portion of the synthetic meat. Members were then randomly selected from the group, and X is the random variable representing the amount of the meat product the member ate (measured in ounces). The probability density function for X is **SCORE: _____ / 35 PTS**

$$f(x) = \begin{cases} k\sin\frac{\pi x}{8}, & x \in [0, 4] \\ 0, & x \notin [0, 4] \end{cases} \quad \text{(for some appropriate constant } k \text{).}$$

[a] Find the probability that a randomly chosen member of the focus group ate at least half the portion of the synthetic meat.
Your final answer must be a number, not an integral.

(4) $\int_0^4 k\sin\frac{\pi x}{8}\,dx = 1$

$k\int_0^4 \sin\frac{\pi x}{8}\,dx = 1$

$k\cdot -\frac{8}{\pi}\cos\frac{\pi x}{8}\Big|_0^4 = 1$ (5)

$k\cdot -\frac{8}{\pi}\left(\cos\frac{\pi}{2} - \cos 0\right) = 1$

$k\cdot -\frac{8}{\pi}(0-1) = 1$

(2) $k = \frac{\pi}{8}$

$P(2 \le X \le 4) = \int_2^4 \frac{\pi}{8}\sin\frac{\pi x}{8}\,dx$ (4)

$= \frac{\pi}{8}\cdot -\frac{8}{\pi}\cos\frac{\pi x}{8}\Big|_2^4$ (2)

$= -\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right)$

$= -\left(0 - \frac{\sqrt{2}}{2}\right)$

$= \boxed{\frac{\sqrt{2}}{2}}$ (3)

[b] Find the median amount of meat product eaten by a member of the focus group.
Your final answer must be a number, not an integral.

$\int_0^{X_{MED}} \frac{\pi}{8}\sin\frac{\pi x}{8}\,dx = \int_{X_{MED}}^4 \frac{\pi}{8}\sin\frac{\pi x}{8}\,dx = \frac{1}{2}$ (4)

$\downarrow$

(2) $\left(-\cos\frac{\pi x}{8}\right)\Big|_{X_{MED}}^4 = \frac{1}{2}$

$-\left(\cos\frac{\pi}{2} - \cos\frac{\pi X_{MED}}{8}\right) = \frac{1}{2}$

(3) $\cos\frac{\pi X_{MED}}{8} = \frac{1}{2}$

$\frac{\pi X_{MED}}{8} = \frac{\pi}{3}$ (2)

$X_{MED} = \frac{8}{3}$ OUNCES

(2) (2)

Find the area of the surface created by revolving the arc of $f(x)=\sqrt{x}-\frac{x^{\frac{3}{2}}}{3}$ on $[1,3]$ about the $\underbrace{x-\text{axis}}_{r=y}$. **SCORE: _____ / 30 PTS**

Your final answer must be a number, not an integral.

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}$ (4)

(3) $2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\right)\sqrt{1+\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}\right)^2}\,dx$ (4) (4)

$= 2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\right)\sqrt{1+\frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x}\,dx$

$= 2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\right)\sqrt{\frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x}\,dx$

$= 2\pi \int_1^3 \left(x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}\right)\left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}}\right)dx$ (4)

$= 2\pi \int_1^3 \left(\frac{1}{2} + \frac{1}{2}x - \frac{1}{6}x - \frac{1}{6}x^2\right)dx$

$= 2\pi \int_1^3 \left(\frac{1}{2} + \frac{1}{3}x - \frac{1}{6}x^2\right)dx$ (4)

$= 2\pi \left(\frac{1}{2}x + \frac{1}{6}x^2 - \frac{1}{18}x^3\right)\Big|_1^3$ (4)

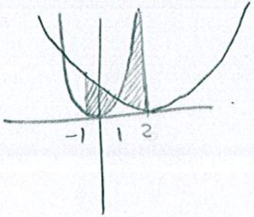
$= \pi \left(x + \frac{1}{3}x^2 - \frac{1}{9}x^3\right)\Big|_1^3$

$= \pi \left(3 + \cancel{3} - \cancel{3} - \left(1 + \frac{1}{3} - \frac{1}{9}\right)\right)$

$= \boxed{\frac{16\pi}{9}}$ (3)

Find the area between the curves $y = 4x^2$ and $y = (x-3)^2$ over the interval $[-1, 2]$. **SCORE: _____ / 25 PTS**

<u>Your final answer must be a number, not an integral. HINT: The answer is NOT 9.</u>



$4x^2 = (x-3)^2$

$4x^2 = x^2 - 6x + 9$

$3x^2 + 6x - 9 = 0$

$3(x+3)(x-1) = 0$

$x = \cancel{-3}, 1$

3 $\int_{-1}^{1} \underbrace{((x-3)^2 - 4x^2)}_{3}\,dx + $ 3 $\int_{1}^{2} \underbrace{(4x^2 - (x-3)^2)}_{3}\,dx$

$= \int_{-1}^{1} \underbrace{(-3x^2 - 6x + 9)}_{1}\,dx + \int_{1}^{2} \underbrace{(3x^2 + 6x - 9)}_{1}\,dx$

$= \underbrace{(-x^3 - 3x^2 + 9x)}_{4}\Big|_{-1}^{1} + \underbrace{(x^3 + 3x^2 - 9x)}_{4}\Big|_{1}^{2}$

$= -1 - 3 + 9 - (1 - 3 - 9) + (8 + 12 - 18 - (1 + 3 - 9))$

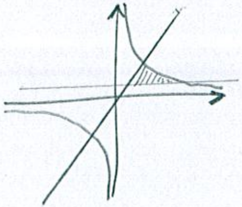
$= 5 \quad - \quad -11 \quad + \quad 2 \quad - \quad -5$

$= \underbrace{23}_{3}$

The region bounded by $y = 2x$, $y = \frac{8}{x}$ and $y = 1$ is revolved about the line $y = 8$. **SCORE: _____ / 40 PTS**

Find the volume of the resulting solid.

<u>Your final answer must be a number, not an integral.</u>



$$3 \quad \underbrace{2\pi}_{6} \boxed{\int_1^4} \underbrace{(8-y)}_{6} \underbrace{\left(\frac{8}{y} - \frac{1}{2}y\right)}_{9} dy$$

$$= 2\pi \int_1^4 \underbrace{\left(\frac{64}{y} - 4y - 8 + \frac{1}{2}y^2\right)}_{6} dy$$

$2x = \frac{8}{x}$

$$= 2\pi \left. \underbrace{\left(64 \ln|y| - 2y^2 - 8y + \frac{1}{6}y^3\right)}_{6} \right|_1^4$$

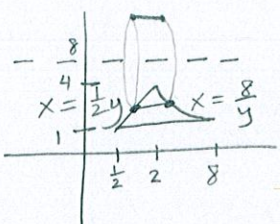
$x^2 = 4$

$x = \pm 2$

$$= 2\pi \left(64(\ln|4| - \ln|1|) - 2(16-1) - 8(4-1) + \frac{1}{6}(64-1)\right)$$

$y = \pm 4$

$$= 2\pi \left(64 \ln 4 - 30 - 24 + \frac{21}{2}\right)$$



$$= \underbrace{\pi(128 \ln 4 - 87)}_{4}$$

OR

ALTERNATE (NOT RECOMMENDED) SOLUTION

$$\underbrace{\boxed{\pi}}_{1\frac{1}{2}}\underbrace{\boxed{\int_{\frac{1}{2}}^{2}}}_{3}\underbrace{\left((8-1)^2-(8-2x)^2\right)}_{7\frac{1}{2}}dx+\underbrace{\boxed{\pi}}_{1\frac{1}{2}}\underbrace{\boxed{\int_{2}^{8}}}_{3}\underbrace{\left((8-1)^2-\left(8-\frac{8}{x}\right)^2\right)}_{7\frac{1}{2}}dx$$

$$=\pi\int_{\frac{1}{2}}^{2}\left(49-(64-32x+4x^2)\right)dx+\pi\int_{2}^{8}\left(49-\left(64-\frac{128}{x}+\frac{64}{x^2}\right)\right)dx$$

$$=\pi\int_{\frac{1}{2}}^{2}\left(\overbrace{\underline{-15+32x-4x^2}}^{3}\right)dx+\pi\int_{2}^{8}\left(\overbrace{\underline{-15+\frac{128}{x}-\frac{64}{x^2}}}^{3}\right)dx$$

$$=\pi\left(\overbrace{\underline{-15x+16x^2-\frac{4}{3}x^3}}^{3}\right)\Big|_{\frac{1}{2}}^{2}+\pi\left(\overbrace{\underline{-15x+128\ln|x|+\frac{64}{x}}}^{3}\right)\Big|_{2}^{8}$$

$$=\pi\left(-30+64-\frac{32}{3}-\left(-\frac{15}{2}+4-\frac{1}{6}\right)-120+128\ln 8+8-(-30+128\ln 2+32)\right)$$

$$=\underline{\pi(128\ln 4-87)}\ 4$$